Solutions

MIIB

S2005 (1102)

1 (a) For $\varepsilon > 0$ $\exists \delta$ such that

$|f(x) - f(a)| < \varepsilon$ for all x such that $|x - a| < \delta$

(or definition of limit $f(x) \to l$ as $x \to a$ together with $l = f(a)$)

$$\frac{f(x) - f(a)}{x - a} \to \text{a limit as } x \to a$$

or $f(x) = f(a) + (x-a)A + (x-a)\eta(x-a)$

where A is constant known as $f'(a)$

and $\eta(x-a) \to 0$ as $x \to a$

or $\frac{f(a+h) - f(a)}{h} \to$ etc. in terms of h

(b) $\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|$

so $f'(x) = 0$ everywhere so (M.V.T.) f is a constant.

(c) Suppose $F(x)$ exists. Then ~~for~~ consider

$G(x) = F(x) - x$, for all $x > 0$ $G'(x) = 1 - 1 = 0$

so $G(x) = C_1$ say (M.V.T could be mentioned here)

Similarly $G(x) = C_2$ say for $x < 0$

and $G(0) = C_3$ say.

So $F(x) = x + C_2$ $x < 0$

$F(0) = C_3$

$F(x) = x + C_1$ $x > 0$

But $F(x)$ is differentiable at 0, so it's continuous at 0 so $C_2 = C_3 = C_1 = C$ say

so $F(x) = x + C$ for all x

but then $F'(x) = 1$ so $F'(0) = f(0) = 1$

which gives contradiction. There is no such function.

$0 < R < \infty$ [3]

(2) ~~For~~ R is said to be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$

if $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x : |x| < R$

if " diverges for $x : |x| > R$

If $\sum_{n=0}^{\infty} a_n x^n$ converges for all x we write $R = \infty$, if $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = 0$ only we write $R = 0$

Suppose

$$|a_n|^{\frac{1}{n}} \to l \text{ as } n \to \infty$$

~~then~~ ~~then~~ we need to show that the series converges for all x such that $|x| < \frac{1}{l}$. Fix such an x. Then then $\exists K$ such that $|x| < \frac{1}{K} < \frac{1}{l}$

then $|a_n x^n| < |a_n| / K^n$

Now $K > l$ so $\exists N$ such that

$$|a_n|^{\frac{1}{n}} < \frac{K+l}{2} \text{ for all } n > N$$

so $|a_n x^n| < \left(\frac{K+l}{2} \cdot \frac{1}{K}\right)^n = \left(\frac{1 + l/K}{2}\right)^n$ convergent geometric series.

[4]

(a) $\sum_{n=0}^{\infty} n x^n$ $\quad n^{\frac{1}{n}} \to 1$ so $l = 1$

(or could use ratio test)

(b) $(n^2)^{\frac{1}{n}} < |a_n|^{\frac{1}{n}} < (n^3)^{\frac{1}{n}}$ so $|a_n|^{\frac{1}{n}} \to n^{\frac{1}{n}}, n^{\frac{2}{n}}, n^{\frac{3}{n}}$
$\to 1, 1, 1$

$l = 1$, Ratio test won't work.

(c) They haven't been given the lim sup version of the n^{th} root test but if somebody uses it that would be the best method.
~~Otherwise~~ Otherwise consider

$5 + 5^2 x^2 + 5^4 x^4 + \ldots + 5^{2n} x^{2n} + \ldots$

~~converges for $x^2 <$~~

or
$5 + 5^2 y + 5^4 y^2 + \ldots 5^{2n} y^n$

$|5^{2n} y|^{\frac{1}{n}} \to 5^2$

so converges for $|x| < \frac{1}{5}$

and ~~...~~ $3x(1 + 3^2 x^2 + \ldots + 3^{2n} x^{2n} + \ldots$

converges for $|x| < \frac{1}{3}$

so the sum converges for $|x| < \frac{1}{5}$

so $R = \frac{1}{5}$

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) (a) If f ~~...~~ and g are:

(1) continuous in an interval (a, b)

(2) Differentiable in $[a, b]$

Then $\exists\, c \in (a, b)$ with

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c)$$

Proof

Let $h(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x)$

for $x \in [a, b]$

then h is continuous in (a, b)
h is differentiable in $[a, b)$

and $h(a) = f(b)g(a) - g(b)f(a) = h(b)$

So by Rolle's Theorem

$c \in (a, b)$ with $h'(c) = 0$

so $(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c)$

L'Hôpital's Rule:

Suppose f, g are differentiable on (a, b)

$f(a+) = g(a+) = 0$, $g'(x) \neq 0$, for all $x \in (a, b)$

then if $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)}$ exists then so does $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)}$ and this equals $\lim \dfrac{f'(x)}{\ldots}$

First look at

$$\lim_{x \to \pi/2} \left(\frac{x - \pi/2}{\cos x} \right) = \lim_{x \to \pi/4} \left(\frac{1}{-\sin x} \right)$$

$$= -1$$

So $$\lim_{x \to \pi/2^-} \left(\frac{x - \pi/2}{\cos x} \right)^4 = -(-1)^4 = -1$$

(h)

(i) $\lim_{x \to 1} \frac{\cos \frac{\pi}{2} x}{x^2 - 1} = \lim_{x \to 1} \frac{-\frac{\pi}{2} \sin \frac{\pi}{2} x}{2x}$

$= -\frac{\pi}{4}$

(ii) $\lim_{x \to 3} \frac{\log(4-x)}{(x^2-9)^{\frac{1}{2}}} = \lim_{x \to 3} \frac{-\frac{1}{4-x}}{\frac{1}{2}(x^2-9)^{-\frac{1}{2}} \cdot 2x}$

$= \lim_{x \to 3} + \frac{1}{x-4} \frac{(x^2-9)^{\frac{1}{2}}}{x} = 0$

(iii) $\cancel{\lim_{x \to \frac{\pi}{2}}} \frac{(x - \frac{\pi}{2})^4}{\cos^2 x - \cot^2 x}$

$= \frac{\sin^2 x \, (x - \frac{\pi}{2})^4}{\cos^2 x (\sin^2 x - 1)}$

$= \frac{\sin^2 x \, (x - \frac{\pi}{2})^4}{-\cos^4 x}$

as $x \to \frac{\pi}{2}$ $\sin^2 x \to 1$

So look at $\lim_{x \to \frac{\pi}{2}} - \frac{(x - \frac{\pi}{2})^4}{\cos^4 x}$

4(i) Taylor's Theorem

Suppose that $f, f', f'', \ldots f^{(n+1)}$ are defined on $[a, x]$ and that $R_{n,a}(x)$ is defined by

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n,a}(x)$$

then (1) $R_{n,a}(x) = \dfrac{f^{(n+1)}(t)}{n!}(x-t)^n(x-a)$

for some $t \in (a, x)$ (Cauchy form)

and (2) $R_{n,a}(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$

for some $\xi \in (a, x)$ (The Lagrange form of remainder)

<u>Proof</u> For $t \in [a, x]$ write

$$f(x) = f(t) + \frac{f'(t)}{1!}(x-t) + \ldots \frac{f^{(n)}(t)}{n!}(x-t)^n + R_{n,t}(x)$$

Consider a, x, n fixed

and write $S(t) = R_{n,t}(x)$, t varying

Then $S(a) = R_{n,a}(x)$ $\quad S(x) = 0$

So $f(x) = f(t) + \frac{f'(t)}{1!}(x-t) + \dots + \frac{f^{(n)}(t)}{n!}(x-t)^n + S(t)$

So $-S(t) = -f(x) + f(t) + \frac{f'(t)}{1!}(x-t) + \dots \frac{f^{(n)}(t)}{n!}(x-t)^n$

and

$$-S'(t) = \cancel{f'(t)} + \left[\cancel{-f'(t)} + \frac{f''(t)(x-t)}{1!}\right]$$
$$+ \left[\cancel{-\frac{f''(t)(x-t)}{1!}} + \frac{f^{(3)}(t)(x-t)}{2!}\right]$$
$$+ \left[-\frac{f^{(n)}}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$

So $$S'(t) = \frac{f^{(n+1)}(t)\ (x-t)^n}{n!}$$

To prove I use M.V.T.

$$\frac{S(x) - S(a)}{x-a} = S'(t) = -\frac{f^{(n+1)}(t)\ (x-t)^n}{n!} \quad \text{for some } t \in (a, x)$$

$$\frac{0 - R_{n,a}(x)}{x-a} = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \quad \text{gives (1)}$$

$\boxed{10}$

Use Cauchy M.V.T on

$F(t)$ and $G(t) = (x-t)^{n+1}$

$$\frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F'(\xi)}{G'(\xi)} \qquad \xi \in (a,x)$$

$$= \frac{\frac{1}{n!} f^{(n+1)}(\xi)\cancel{(x-\xi)^n}}{-(n+1)\cancel{(x-\xi)^n}}$$

$$= \frac{0 - R_{n,a}(x)}{-(x-a)^{n+1}} = \frac{\cancel{f^{(n+1)}(\xi)(x-\xi)^n}}{\cancel{(n+1)!}}$$

$$= \frac{1}{(n+1)!} f^{(n+1)}(\xi)$$

so $R_{n,a}(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-a)^{n+1}$

proves (2)

(b)

$f(x) = \log(1+x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -1.(1+x)^{-2}$ $\qquad f''(0) = -1$

$f'''(x) = -1.-2.(1+x)^{-3}$ $\qquad f'''(0) = 2$

$f^{(iv)}(x) = -1.-2.-3(1+x)^{-4}$ $\qquad f^{iv}(0) = -3!$

$\vdots$

$f^{(n)}(x) = (-1)^{n+1}(n-1)!(1+x)^{-n}$ $\qquad f^{(n)}(0) = (-1)^{n+1}(n-1)!$

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots (-1)^{n+1}\frac{x^n}{n} + \dots$$

(ii) $\log(1+x^2)$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \dots + (-1)^{n+1}\frac{x^{2n}}{n} \dots$$

(iii) $\log(1+x+x^2+x^3)$

$$= \log\left(\frac{1-x^4}{1-x}\right) = \log(1-x^4) - \log(1-x)$$

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$$\log(1-x) = \sum_{n=1}^{\infty} (-1)^{n+1}(-1)^n \frac{x^n}{n}$$

$$= -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\log(1-x^4) = -\sum_{n=1}^{\infty} \frac{x^{4n}}{n}$$

So $\log(1+x+x^2+x^3) = +\sum_1 \frac{x^n}{n} - \sum_{n=1}^{\infty} \frac{x^{4n}}{n}$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8}\right.$$

$$- \frac{x^4}{1} + \ldots - \frac{x^8}{2} + \ldots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^7}{7} - \frac{3x^8}{9} + \frac{x^{12}}{17}$$

$$- \frac{x^{19}}{19} - \frac{3x^{20}}{20}$$

5. A partition P is a finite set of points

$a = x_0 < x_1 < \ldots < x_n = b$

let $M_i = \sup\{f(x),\ x_{i-1} \le x \le x_i\}$
$m_i = \inf\{f(x),\ x_{i-1} \le x \le x_i\}$ $\Big\}\ i = 1, \ldots n$
and $\Delta_i x = x_i - x_{i-1}$

The $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$

[13]

~~Such~~ Write $\inf_P U(P,f)$ for the ② inf of $U(P,f)$ taken over all partitions P and $\sup_P L(P,f)$ ① the supremum of $L(P,f)$ taken over all P (exist since f is bounded) then f is Riemann integrable if ① = ②.

We use following Theorem:

f is Riemann integrable on $[a,b]$ iff $\forall \varepsilon > 0$ there exists partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$

W.log assume f monotonic increasing then let P_n be given by $x_i = a + \frac{i}{n}(b-a)$ $i = 0, 1, \ldots n$.

$$U(P_n,f) - L(P_n,f)$$

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

$$= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\frac{b-a}{n}$$

$$= \frac{(b-a)}{n}(f(b) - f(a))$$

Given $\varepsilon > 0$ choose N so that

$$\frac{(b-a)}{N}(f(b) - f(a)) < \varepsilon.$$

6. If f is Riemann integrable and if there is a differentiable F on $[a,b]$ such that $F'(x) = f(x)$ $\forall x \in [a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Proof</u> Let $\varepsilon > 0$ be given.

Choose P

$a = x_0 < x_1 < \dots < x_n = b$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

M.V.T. gives $t_i \in [x_{i-1}, x_i]$ $i = 1, \dots n$

$$F(x_i) - F(x_{i-1}) = f(t_i)\,\Delta x_i$$

So
$$\sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

$$= F(b) - F(a)$$

As
$$L(P,f) \le \sum_{i=1}^{n} f(t_i)\,\Delta x_i \le U(P,f)$$

and
$$L(P,f) \le \int_a^b f(x)\,dx \le U(P,f)$$

we have $\left| F(b) - F(a) - \int_a^b f(x) \right| < \varepsilon$ for all $\varepsilon > 0$

so $\int_a^b f(x)\,dx = F(b) - F(a)$

(15)

Suppose $F'(x) = \dfrac{1}{\exp(x)+1}$

(a)

then $G(x) = F(x) - F(1)$

$$G'(x) = F'(x) = \frac{1}{\exp(x)+1}$$

(b) $G(x) = F(x^2) - F(x)$

$$G'(x) = F'(x^2)\,2x - F'(x)$$

$$= \frac{2x}{\exp(x^2)+1} - \frac{1}{\exp(x)+1}$$

(c) $G(x) = \displaystyle\int_1^{F(x)-F(1)} \frac{1}{\exp(t)+1}\,dt$

$$= F(F(x)-F(1)) - F(1)$$

$$G'(x) = F'(F(x)-F(1))\cdot F'(x)$$

$$= \left(\frac{1}{\exp\left(\int_1^x \frac{1}{\exp t+1}\right)+1}\right)\cdot\frac{1}{\exp(x)+1}$$

We need to find (for the first term:)

$$\int_1^x \frac{dt}{e^t+1}$$

$$\frac{1}{e^t+1} = \frac{e^t - e^t + 1}{e^t+1}$$

$$= 1 - \frac{e^t}{e^t+1}$$

So

$$\int_1^x \frac{dt}{e^t+1} = \Big[t - \log(e^t+1) \Big]_1^x$$

$$= x - 1 - \log(e^x+1) + \log(e+1)$$

So

$$\exp\left(\int_1^x \frac{dt}{e^t+1}\right) = e^{x-1+\log\left(\frac{e+1}{e^x+1}\right)}$$

$$= e^{x-1}\left(\frac{e+1}{e^x+1}\right)$$

$$G'(x) = \frac{1}{e^{x-1}\left(\frac{e+1}{e^x+1}\right)+1} \qquad \frac{1}{e^x+1}$$

$$= \frac{1}{e^{x-1}(e+1) + e^x + 1} = \frac{1}{2e^x + e^{x-1} + 1}$$

$$= \frac{e}{e + e^x + 2e^{(x+1)}}$$